Problem 5C,4
Show that Lebesgue meausure on $\mathbb{R}^n$ is translation invariant.

Proof. This is a standard method. We can first check the result holds for rectangles, which is easy and obvious. Then we can check the result holds for finite disjoint union of rectangles. Finally we can use these rectangles to approximate general open sets and Borel sets. □

Problem 5C,5
Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{B}_n$-measurable and $t \in \mathbb{R} - \{0\}$.Define $f_t : \mathbb{R}^n \to \mathbb{R}$ by $f_t(x) = f(tx)$.

- Prove that $f_t : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{B}_n$-measurable.
- Prove that if $\int_{\mathbb{R}^n} f(x)d\lambda_n$ is defined,

$$\int_{\mathbb{R}^n} f_t(x)d\lambda_n = \frac{1}{|t|^n}\int_{\mathbb{R}^n} f(x)d\lambda_n$$

Proof.
- Just note that $f_t^{-1}(B) = \frac{1}{t}f^{-1}(B)$.
- Obviously the equality holds for characteristic function of rectangles, and then use a limiting argument it also holds for any characterisk function and so for simple functions. The for any integrable nonnegative function, we can find a sequence of simple functions, which converges pointwise and monotone to the given function. Then by monotone convergence theorem, the result holds for any integrable nonnegative function. For general function, just splits it into positive part and negative part. This standard argument completes the proof.
□

Problem 6A,4
Suppose V is a metric space.

- Prove that the any union of open subsets is open subset of V.
- Prove that finite intersection of open subsets is open subset of V.

Proof.
- Let $U_i, i \in I$ are open sets, where I is an index set. Let $U = \bigcup_{i\in I} U_i$. We need to check that U is open. Let any $x \in U$, then there exists i_0 such that $x \in U_{i_0}$. So there exists positive number r such that $B(x, r) \subset U_{i_0}$, thus $B(x, r) \subset U$. By definition, U is an open set.
- Let $U_i.i = 1, 2, \dots, k$ be a finite set of open subset in v. Let $U = \bigcap_{i=1}^{k} U_i$ be their intersection. For any $x \in U$, there exist positive number r_i such that $B(x, r_i) \subset U_i$. Then let $r = \min r_1, r_2, \dots, r_k$, so $B(x, r) \subset U$, which implies that U is open.
□

Problem 6A,5
Suppose V is a metric space.

- Prove that the any intersection of closed subsets is closed subset of V.
- Prove that finite union of closed subsets is closed subset of V.

Proof. This followsdirectly from Problem 6A,4. □

Problem 6A,6

- Prove that if v is a metric space, $f \in V, r > 0$, then $B(\bar{f}, r) \subset \bar{B}(f, r)$.

- Give an example of a metric space V, $f \in V, r > 0$, and $B(\bar{f}, r) \neq \bar{B}(f, r)$.

Proof.
- Note that $\bar{B}(f,r)$ is closed, and $B(f,r)$ is an open subset of $\bar{B}(f,r)$. So by definition, we get that $B(\bar{f},r) \subset \bar{B}(f,r)$.
- Let S be any set and consider the discrete distance on S, which by definition is $d(x,y) = 1, if x \neq y$, $d(x,x) = 0$. So we know that for any $x \in S$, $B(x,1) = \{x\}$ is closed but $\bar{B}(x,1) = S$.
□

Problem 6B,6
Suppose $(X, \mathcal{S})$ is a measurable space and $f, g : X \to \mathbb{C}$ is an $\mathcal{S}$-measurable function. Prove that

- $f + g, f - g, fg$ are $\mathcal{S}$-measurable functions.
- If $g(x) \neq 0$, then $\frac{f}{g}$ is $\mathcal{S}$-measurable function.

Proof.
- By definition, we only need to care about the real case. Note that

$$\{x|f(x) + g(x) > t\} = \bigcup_i (\{x|f(x) > r_i\} \bigcap \{x|g(x) > t - r_i\})$$

where $\{r_i\}$ is all rational number. From this formula, we know that $f+g$ is measurable. For $f-g$, just note that $-g$ is measurable and then use previous result. For fg, just use similar formulas to show that f^2 is measurable and then use $fg = \frac{1}{4}((f+g)^2 - (f-g)^2)$.

- Just note that $\frac{1}{g}$ is measurable and then use previous result.
□

Problem 6B,7
Suppose $(X, \mathcal{S})$ is a measurable space and $f_1, f_2, \ldots$ is a sequence of measurable functions from X to $\mathbb{C}$. Suppose $\lim_{k\to\infty} f_k(x)$ exists for any $x \in X$. Define

$$f(x) = \lim_{k\to\infty} f_k(x)$$

Prove that f is an $\mathcal{S}$-measurable function.

Proof. As the previous problem, we only need to care about the real case. We first claim that for any sequence of function $f_1, f_2, \ldots$ the following are also measurable:

$$\sup_{k\geq 1} f_k(x), \inf_k f_k(x).$$

In fact, for any $t \in \mathbb{R}$, we have

$$\{x| \sup_{k\geq 1} f_k(x) > t\} = \bigcup_k \{x|f_k(x) > t\}$$

$$\{x| \inf_{k\geq 1} f_k(x) > t\} = \bigcap_k \{x|f_k(x) > t\}$$

which implies our claim is right. Note that

$$\limsup_{k\to\infty} f_k(x) = \inf_k \sup_{i>k} f_i(x)$$

$$\liminf_{k\to\infty} f_k(x) = \sup_k \inf_{i>k} f_i(x)$$

which implies that $\limsup_{k\to\infty} f_k, \liminf_{k\to\infty} f_k$ are also measurable. So return back to our situation, obviously we have $f(x) = \limsup_{k\to\infty} f_k$ is also measurable. □